The reflexivity of hyperexpansions and their Cauchy dual operators.

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Theorem (Von Neumann)
Let $T \in \mathcal{B}(\mathcal{H})$ be normal. If $SP = PS$ for any projection P such that $TP = PT$, then S, belongs to weakly close star algebra generated by T and identity.

Sarason and Reflexivity

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Theorem (Sarason-1966)
Let $T \in \mathcal{B}(\mathcal{H})$ be normal.
If $PSP = SP$ for any projection P such that, $PTP = TP$, then $S \in \mathscr{W}(T)$, ♣
where $\mathscr{W}(T)$ is the weakly closed algebra generated by T and identity.

Stronger hypothesis, and stronger conclusion!!

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Theorem (Sarason-66)
Let A and B be analytic Toeplitz operators, and if B leaves invariant every invariant subspace of A, then B belongs to the weakly closed algebra generated by A

Let, the collection of all invariant subspaces of T is denoted by $\textbf{Lat}\,T$.

Definition
Let $\mathscr{W}(T)$ be WOT-closed sub-algebra of $\mathcal{B}(\mathcal{H})$ generated by T and identity I.

$$\textbf{Lat}\,\mathscr{W}(T) = \bigcap_{S \in \mathscr{W}(T)} \textbf{Lat}\,S$$

Invariant Subspaces, Algebra and Lattices

Definition
Let $\mathscr{F}$ be any collection of subspaces of $\mathcal{H}$, then define

$$\textbf{Alg}\mathscr{F} = \{A \in \mathcal{B}(\mathcal{H}) : \mathscr{F} \subset \textbf{Lat}A\}.$$

Definition (Reflexivity- Rajdavi and Rosenthal 1968)
An operator $T \in \mathcal{B}(\mathcal{H})$ is called reflexive if

$$\textbf{AlgLat}\mathscr{W}(T) = \mathscr{W}(T).$$

Remark
For every $T \in \mathcal{B}(\mathcal{H})$,

$$\mathscr{W}(T) \subset \textbf{AlgLat}T.$$

Examples of reflexive operators

- Sarason-1966: Normal operators are reflexive.
- Olin & Thomson-1982: Sub-normal operators are reflexive.
- Sarason-1966: Unilateral shift on multiplicity one is reflexive.
- Deddens-1971: Every isometry is reflexive.
- Sheild-1974: If T an injective unilateral weighted shift, such that T^* has a non zero eigenvalue, then T is reflexive.

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- Sheild-1974: If T an injective unilateral weighted shift, such that T^* has a non zero eigenvalue, then T is reflexive.
- Let $\mathcal{H}$ be a scalar reproducing kernel Hilbert space, then the multiplier algebra $\mathcal{M}(\mathcal{H})$ is reflexive.

Reproducing kernel Hilbert space

Definition (Reproducing Kernel Hilbert space)
Let $\mathcal{H} \subset \mathcal{F}(\triangle, \mathbb{C})$ be a complex separable Hilbert space.

We say $\mathcal{H}$ is a reproducing kernel Hilbert space (rkHs), if for every $w \in \triangle$, there exists $c_w \geq 0$

$$|f(w)| \leq c_w \|f\|_{\mathcal{H}}, \quad \text{for all } f \in \mathcal{H}.$$

Equivalently, there exits parametrized family $\{k_w\}_{w \in \triangle} \subset \mathcal{H}$, such that,

$$f(w) = \langle f,\, k_w \rangle.$$

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$$\mathcal{M}(\mathcal{H}) := \{\phi \in \mathcal{F}(\triangle, \mathbb{C}) : f \in \mathcal{H} \text{ implies } \phi f \in \mathcal{H}\}.$$

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$\mathcal{M}(\mathcal{H})$ is unital weakly closed reflexive algebra.

CNP kernel

Definition
A reproducing kernel is called complete Nevanlinna Pick type:
Given $n \in \mathbb{Z}_{+}$, if an operator $R : V(:= sp\{k_{w_1}, \ldots, k_{w_n}\}) \to V$

$$R(k_{\lambda_i}) = \lambda_i k_{w_i},$$

is a contraction, then there exists a $\phi \in \mathcal{M}(\mathcal{H})$ such that M_ϕ is a contraction and $M_\phi^*|_V = R$.

Super reflexive multiplier algebra

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Theorem (Davidson & Hamilton-2011)
For a CNP kernel k, the multiplier algebra $\mathcal{M}(\mathcal{H}_k)$ is super reflexive.

Definition
For $T \in \mathcal{B}(\mathcal{H})$ define

$$B_n(T) := \sum_{p=0}^{n} (-1)^p \binom{n}{p} T^{*p} T^p. \tag{1}$$

(i) T is said to be *completely hyperexpansive* if $B_n(T) \leqslant 0$ for all $n \in \mathbb{N}$.

(ii) For $m \in \mathbb{N}$, T is said to be *m-hyperexpansive* if $B_n(T) \leqslant 0$ for $n = 1, \ldots, m$.

(iii) T is said to be *m-isometric* if $B_m(T) = 0$, for some $m \in \mathbb{N}$.

Completely hyper-expansive operators

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Example

(1) An 2-isometry is completely hyperexanpsive.

(2) An isometry is completely hyperexpansive.

Definition (Dirichlet space)
Let $\mathcal{D}$, denote the space of analytic functions on a Hilbert space of analytic functions defined as follows:

$$\mathcal{D} := \{f \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : D(f) < \infty\}$$

where

$$D(f) := \int_{\mathbb{D}} |f'(\zeta)|^2 dA(\zeta), \tag{2}$$

and dA, is the normalized area measure on $\mathbb{D}$. Let M_z, denote the multiplication by coordinate function

$$(M_z f)\omega = \omega f(w), \quad \omega \in \triangle.$$

- M_z is cyclic 2-isometry, and hence completely hyperexpansive.

Theorem
Let $T \in \mathcal{B}(\mathcal{H})$ is an analytic (i.e. $\cap_n T^n(\mathcal{H}) = 0$), completely hyperexpansive operator, such that $\dim(\mathcal{H} \ominus T(\mathcal{H})) = 1$. Then there exists a positive Borel measure μ on $\overline{\mathbb{D}}$ such that T is unitarily equivalent to M_z on

$$\mathcal{D}_\mu := \{f \in \mathcal{O}(\mathbb{D}, \mathbb{C}) : \|f\|_\mu < \infty\}$$

where

$$\|f\|_\mu^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 U_\mu(\zeta) dm(\zeta), \tag{3}$$

and dm is the normalized area measure on $\mathbb{D}$. Further

$$U_\mu(\zeta) := \int_{\mathbb{D}} log\left|\frac{1-\overline{z}\zeta}{\zeta - z}\right| \frac{d\mu(z)}{1-|z|^2} + \int_{\partial\mathbb{D}} \frac{1-|\zeta|^2}{|z-\zeta|^2} d\mu(z), \quad \zeta \in \mathbb{D}. \tag{4}$$

Reflexivity of hyperexpansive operators

Theorem (S. Podder, —)
Let $T \in B(\mathcal{H})$ be completely hyperexpansive with $\dim(\mathcal{H} \ominus T(\mathcal{H})) = 1$. Then T is super-reflexive.

Corollary

If T is a cyclic 2-isometry in $B(\mathcal{H})$, then T is super-reflexive.

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